Int. J. Solids Structures, 1968, Vol. 4, pp. 1067 to 1080. Pergamon Press. Printed in Great Britain

BENDING OF A SIMPLY SUPPORTED CIRCULAR CYLINDRICAL SHELL SUBJECTED TO UNIFORM LINE LOAD ALONG A GENERATOR

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Abstract—The complex differential equation of cylindrical shells, given by Novozhilov [1], is used to obtain the state of stress for a simply supported closed thin circular cylindrical shell acted upon by a uniform inward radial line load along a generator. The problem is solved by obtaining a closed form particular integral of the differential equation, and satisfying the edge conditions with the aid of complementary solutions in the form of a single Fourier series which converges very rapidly for the region near the middle of the shell. For comparatively long shells an approximate expression, in closed form, is derived for the region far away from the edges. A mathematical proof of the convergence of the series is given, and numerical results for several ratios of length to radius are presented.

1. INTRODUCTION

DURING the past twenty years, several authors have investigated the problem of bending of a circular cylindrical shell, with finite length, under the action of a discontinuous surface load. Odqvist [2], Hoff *et al.* [3] and Cooper [4] have analysed the problem of a simply supported circular cylindrical shell subjected to a radial line load along a generator. They employed the same method of approach, but different theories of cylindrical shells, to obtain an approximate solution in the form of a single Fourier series for any type of local loadings of the shell. Meanwhile Bijlaard [5] investigated a similar problem by representing the solution in the form of double Fourier series. Subsequently Meck [6] solved the problem of bending of a circular cylindrical shell under a varying circumferential line load by reducing the standard eight order differential equation to two fourth-order equations. Nash and Bridgland [7] also have investigated the problem of local loading of a circular cylindrical shell by employing finite Fourier transform technique. However, in all the above works the primary difficulty is slow convergence of the series, especially near the points where the stresses are critical. This is obviously due to the fact that the function representing the discontinuous load, is expanded into infinite series.

However, in the present investigation, a closed form particular integral for the discontinuous load is found, and hence the series representing the homogeneous solution converges rapidly. It should also be mentioned here that the governing equations used by the previous authors are either unsuitable, or too complicated for the present problem compared to the method employed in this investigation. For example, Donnell's equation used in [3] is unsuitable for closed cylindrical shells, and Flügge's equation used in [5–7] leads to cumbersome and lengthy expressions if the present technique is employed. Novozhilov's theory of thin cylindrical shells, on the other hand, incorporates various theories of cylindrical shells into one complex equation, which can be used for closed or open shells of any length [1], and yet it is not too complicated for the method of solution employed in the present work.

2. FORMULATION

The complex differential equation of cylindrical shells, given by Novozhilov [1], is in the form

$$\Delta\Delta\tilde{T}+\frac{\partial^2\tilde{T}}{\partial\theta^2}+2b^2i\frac{\partial^2\tilde{T}}{\partial\xi^2} = 2b^2Ri\Delta p, \tag{1}$$

where $\tilde{T}$ is the complex potential, R is the radius of the middle surface, and p is the radial external load per unit area of the middle surface taken positive outward. Other quantities and symbols in equation (1) are defined by:

$$\begin{aligned} &\Delta \equiv \frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\theta^2}, \qquad b^2 = \sqrt{[3(1-\nu^2)]}\frac{R}{t}, \\ &\xi = \frac{x}{R}, \qquad \theta = \frac{y}{R}, \qquad i = \sqrt{(-1)}. \end{aligned} \tag{2}$$

Here t is the thickness of the shell, ν is Poisson's ratio, and x and y are respectively the actual distances on the middle surface measured along axial and circumferential directions.

The resultant forces T_ξ, T_θ, $T_{\xi\theta}$, $T_{\theta\xi}$, resultant moments M_ξ, M_θ, $M_{\xi\theta}$, $M_{\theta\xi}$ and transverse shears N_ξ, N_θ are obtained from the following relations [1]

$$\left.\begin{aligned} \tilde{T}_1 &= T_\xi - i\frac{2b^2(M_\theta-\nu M_\xi)}{R(1-\nu^2)}, \\ \tilde{T}_2 &= T_\theta - i\frac{2b^2(M_\xi-\nu M_\theta)}{R(1-\nu^2)}, \\ \tilde{S} &= S+i\frac{2b^2H}{R(1-\nu)}, \\ N_\xi &= \frac{1}{R}\left(\frac{\partial M_\xi}{\partial\xi}+\frac{\partial M_{\theta\xi}}{\partial\theta}\right), \\ N_\theta &= \frac{1}{R}\left(\frac{\partial M_{\xi\theta}}{\partial\xi}+\frac{\partial M_\theta}{\partial\theta}\right), \end{aligned}\right\} \tag{3}$$

where

$$\left.\begin{aligned} S &= T_{\xi\theta}-\frac{M_{\theta\xi}}{R} = T_{\theta\xi}, \\ H &\approx M_{\xi\theta} \approx M_{\theta\xi}, \end{aligned}\right\} \tag{4}$$

and $\tilde{T}_1$, $\tilde{T}_2$, and $\tilde{S}$ are related to the complex potential as follows [1]

$$
\left.\begin{aligned}
\tilde{T}_1 &= \tilde{T}-\frac{i}{2b^2}\Delta\tilde{T}-Rp, & \frac{\partial\tilde{S}}{\partial\xi} &= -\frac{i}{2b^2}\left(\frac{\partial\,\Delta\tilde{T}}{\partial\theta}+\frac{\partial\tilde{T}}{\partial\theta}\right)-R\frac{\partial p}{\partial\theta},\\
\tilde{T}_2 &= \tilde{T}-\tilde{T}_1, & \frac{\partial\tilde{S}}{\partial\theta} &= -\frac{\partial\tilde{T}}{\partial\xi}+\frac{i}{2b^2}\frac{\partial\Delta\tilde{T}}{\partial\xi}+R\frac{\partial p}{\partial\xi}.
\end{aligned}\right\} \tag{5}
$$

The positive directions of the above resultant forces and moments are the same as those given in [1].

The displacements u, v, and w of the middle surface respectively along the ξ, θ, and z directions, where z is the axis normal to the middle surface taken positive outward, are related to the resultant forces and moments as follows [1]

$$
\left.\begin{aligned}
M_\xi-\nu M_\theta &= -\frac{Et^3}{12R^2}\frac{\partial^2 w}{\partial\xi^2},\\
M_\theta-\nu M_\xi &= -\frac{Et^3}{12R^2}\left(\frac{\partial^2 w}{\partial\theta^2}-\frac{\partial v}{\partial\theta}\right),\\
T_\xi-\nu T_\theta &= \frac{Et}{R}\frac{\partial u}{\partial\xi},\\
T_\theta-\nu T_\xi &= \frac{Et}{R}\left(\frac{\partial v}{\partial\theta}+w\right),\\
S &= \frac{Et}{2R(1+\nu)}\left(\frac{\partial v}{\partial\xi}+\frac{\partial u}{\partial\theta}\right),\\
H &= \frac{Et^3}{12R^2(1+\nu)}\left(-\frac{\partial^2 w}{\partial\xi\partial\theta}+\frac{\partial v}{\partial\xi}\right).
\end{aligned}\right\} \tag{6}
$$

Here E is the modulus of elasticity.

If the origin is chosen at the middle of the cylinder, one must have, for a simply supported shell, the following edge conditions [1]

$$
v = w = T_\xi = M_\xi = 0 \qquad \text{at } \xi = \pm l/R, \tag{7}
$$

where l is half the length of the cylindrical shell. These conditions imply, considering the second and the fourth of equations (6), that $T_\xi = T_\theta = M_\xi = M_\theta = 0$ at the supports. Therefore, from (3) and (5), we have

$$
\tilde{T} = \tilde{T}_2 = 0 \qquad \text{at } \xi = \pm l/R. \tag{8}
$$

Hence apart from rigid body displacements which do not affect the stresses, the conditions in (8) are equivalent to those of (7).

3. SOLUTION

Let us assume that a simply supported circular cylindrical shell is acted upon, along a generator, by a uniform inward radial line load of intensity $\bar{p}$ per unit length of the shell. Choosing the origin at the middle of the shell, and along this line load, one could represent the normal pressure p in the form

$$p = -\frac{\bar{p}}{R}\delta(\theta) \equiv \frac{\bar{p}}{R}\left[-\delta(\theta)+\frac{1}{\pi}\cos\theta\right]-\frac{\bar{p}}{\pi R}\cos\theta, \qquad \pi \geq \theta \geq -\pi, \tag{9}$$

where $\delta(\theta)$ is the unit impulse function.

The terms in the bracket in the right-hand side of (9) correspond to a self equilibrated loading which does not cause any total bending of the shell in the plane of the line load, while the effect of the last term is the bending of the shell similar to that of a beam. Substituting (9) in (1) we obtain

$$\Delta\Delta\tilde{T}+\frac{\partial^2\tilde{T}}{\partial\theta^2}+2ib^2\frac{\partial^2\tilde{T}}{\partial\xi^2} = 2ib^2\bar{p}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left[-\delta(\theta)+\frac{1}{\pi}\cos\theta\right]+2i\frac{b^2\bar{p}}{\pi}\cos\theta. \tag{10}$$

The general solution of (10) is sought in the form

$$\tilde{T} = \psi(\theta)+F(\xi)\cos\theta+\sum_{m=0}^{\infty}\phi_m(\xi)\cos m\theta, \tag{11}$$

in which the series on the right-hand side represents the complementary solutions of (10). $\psi(\theta)$ and $F(\xi)$ are, respectively, particular integrals of

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\theta^2}+\psi = 2ib^2\bar{p}\left[-\delta(\theta)+\frac{1}{\pi}\cos\theta\right], \tag{12}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[\frac{\mathrm{d}^2F}{\mathrm{d}\xi^2}-2(1-ib^2)F\right] = 2i\frac{b^2\bar{p}}{\pi}. \tag{13}$$

In order to find a particular integral of (12), we proceed with the known method of variation of parameters [8] to get

$$\psi(\theta) = 2ib^2\bar{p}\left\{-\cos\theta\int\left[-\delta(\theta)+\frac{1}{\pi}\cos\theta\right]\sin\theta\,\mathrm{d}\theta+\sin\theta\int\left[-\delta(\theta)+\frac{1}{\pi}\cos\theta\right]\cos\theta\,\mathrm{d}\theta\right\}. \tag{14}$$

Taking now into consideration that ψ must be an even function of θ, we arrange the constants of integration so that:

$$\int\delta(\theta)\,\mathrm{d}\theta = U(\theta), \qquad \int U(\theta)\,\mathrm{d}\theta = K(\theta), \tag{15}$$

where

$$U(\theta) = \tfrac{1}{2}\varepsilon(\theta), \qquad K(\theta) = \tfrac{1}{2}\theta\varepsilon(\theta),$$

$$\varepsilon(\theta) = \begin{cases} + & \text{for } \theta > 0, \\ - & \text{for } \theta < 0. \end{cases}$$

Using (15), the right-hand side of (14) is evaluated by integration by parts. After a few simplifications, we find

$$\psi(\theta) = -\varepsilon(\theta)b^2\bar{p}i\sin\theta+\frac{ib^2\bar{p}}{2\pi}(\cos\theta+2\theta\sin\theta), \qquad \pi \geq \theta \geq -\pi. \tag{16}$$

It is seen that ψ is a continuous function of θ, as it should be, in the region prescribed. Physically ψ is the solution for an infinitely long shell in which the stresses do not depend on ξ as the result of the action of a uniform line load equilibrated by a distributed sinusoidal load.

A particular integral of (13) is obtained in the usual way, and hence one has

$$F(\xi) = -\frac{b^2 i\bar{p}}{2\pi(1-ib^2)}\xi^2. \tag{17}$$

In order to find the complementary solution of (10), we substitute the general term of the series $\phi_m(\xi)\cos m\theta$ into the homogeneous equation (10) to get

$$\frac{d^4\phi_m}{d\xi^4}+(2ib^2-2m^2)\frac{d^2\phi_m}{d\xi^2}+(m^4-m^2)\phi_m = 0, \qquad m = 0, 1, 2, \ldots. \tag{18}$$

The solution of (18), considering that ϕ_m must be an even function of ξ, is obtained in the usual way. Therefore we have

$$\left.\begin{aligned} \phi_0 &= A_0+C_0\cosh\alpha_0\xi,\\ \phi_1 &= A_1+C_1\cosh\alpha_1\xi,\\ \phi_m &= A_m\cosh\beta_m\xi+C_m\cosh\alpha_m\xi, \end{aligned}\right\} \tag{19}$$

in which

$$\begin{aligned} \alpha_0 &= b(1-i),\\ \alpha_1 &= [2(1-ib^2)]^{\frac{1}{2}},\\ \alpha_m &= \{\tfrac{1}{2}(m^2+\gamma)+\tfrac{1}{2}[(m^2+\gamma)^2+(\lambda+b^2)^2]^{\frac{1}{2}}\}^{\frac{1}{2}}-i\{-\tfrac{1}{2}(m^2+\gamma)+\tfrac{1}{2}[(m^2+\gamma)^2+(\lambda+b^2)^2]^{\frac{1}{2}}\}^{\frac{1}{2}},\\ &\qquad\qquad m = 2, 3, 4, \ldots\\ \beta_m &= \{\tfrac{1}{2}(m^2-\gamma)+\tfrac{1}{2}[(m^2-\gamma)^2+(\lambda-b^2)^2]^{\frac{1}{2}}\}^{\frac{1}{2}}+i\{-\tfrac{1}{2}(m^2-\gamma)+\tfrac{1}{2}[(m^2-\gamma)^2+(\lambda-b^2)^2]^{\frac{1}{2}}\}^{\frac{1}{2}},\\ \gamma &= \{\tfrac{1}{2}(m^2-b^4)+\tfrac{1}{2}[(m^2-b^4)^2+4m^4b^4]^{\frac{1}{2}}\}^{\frac{1}{2}},\\ \lambda &= \{-\tfrac{1}{2}(m^2-b^4)+\tfrac{1}{2}[(m^2-b^4)^2+4m^4b^4]^{\frac{1}{2}}\}^{\frac{1}{2}}. \end{aligned}$$

In equation (19), A_0, C_0, A_1, C_1, A_m, and C_m are unknown complex constants to be determined from the edge conditions at $\xi = \pm l/R$.

We now substitute (16), (17) and (19) into (11) to obtain

$$\begin{aligned} \tilde{T} = &-\varepsilon(\theta)b^2\bar{p}i\sin\theta+\frac{ib^2\bar{p}}{2\pi}(\cos\theta+2\theta\sin\theta)-\frac{ib^2\bar{p}}{2\pi(1-ib^2)}\xi^2\cos\theta\\ &+A_0+C_0\cosh\alpha_0\xi+(A_1+C_1\cosh\alpha_1\xi)\cos\theta\\ &+\sum_{m=2}^{\infty}(A_m\cosh\beta_m\xi+C_m\cosh\alpha_m\xi)\cos m\theta, \qquad \pi \geq \theta \geq -\pi. \end{aligned} \tag{20}$$

Having determined the complex potential $\tilde{T}$ one could find all the resultant forces and resultant moments from relations (5), (3) and (4). For the sake of brevity, we shall give only the expressions for $\tilde{T}_1$ and $\tilde{T}_2$.

$$\begin{aligned}\tilde{T}_1 = & -\varepsilon(\theta)\bar{p}(b^2 i-\tfrac{1}{2})\sin\theta+\frac{\bar{p}}{2\pi}(\cos\theta+2\theta\sin\theta)(ib^2-\tfrac{1}{2})+A_0\\ & +\left\{-\frac{i}{2b^2}\left[-A_1+(\alpha_1^2-1)C_1\cosh\alpha_1\xi+\frac{b^2\bar{p}i(\xi^2-2)}{\alpha_1^2\pi}\right]+\frac{\bar{p}}{\pi}+A_1\right.\\ & \left.+C_1\cosh\alpha_1\xi-\frac{b^2\bar{p}i}{\alpha_1^2\pi}\xi^2\right\}\cos\theta+\sum_{m=2}^{\infty}\left\{C_m\cosh\alpha_m\xi+A_m\cosh\beta_m\xi\right.\\ & \left.-\frac{i}{2b^2}[(\alpha_m^2-m^2)C_m\cosh\alpha_m\xi+(\beta_m^2-m^2)A_m\cosh\beta_m\xi]\right\}\cos m\theta,\\ & \pi\geq\theta\geq-\pi,\end{aligned}\tag{21}$$

$$\begin{aligned}\tilde{T}_2 = & -\varepsilon(\theta)\frac{\bar{p}}{2}\sin\theta+\frac{\bar{p}}{4\pi}(\cos\theta+2\theta\sin\theta)+\frac{i}{2b^2}C_0\alpha_0^2\cosh\alpha_0\xi\\ & +\left\{\frac{i}{2b^2}\left[-A_1+(\alpha_1^2-1)C_1\cosh\alpha_1\xi+\frac{ib^2\bar{p}(\xi^2-2)}{\alpha_1^2\pi}\right]-\frac{\bar{p}}{\pi}\right\}\cos\theta\\ & +\frac{i}{2b^2}\sum_{m=2}^{\infty}[(\alpha_m^2-m^2)C_m\cosh\alpha_m\xi+(\beta_m^2-m^2)A_m\cosh\beta_m\xi]\cos m\theta,\\ & \pi\geq\theta\geq-\pi.\end{aligned}\tag{22}$$

It remains to evaluate A_0, C_0, A_1, C_1, A_m and C_m from the edge conditions at $\xi = \pm l/R$. To achieve this, we expand the first two terms of the right-hand sides of (20) and (22) in Fourier series and apply conditions (8) to get

$$\begin{aligned}&\frac{ib^2\bar{p}}{\pi}+\frac{ib^2\bar{p}}{\alpha_1^2\pi}\frac{l^2}{R^2}\cos\theta-\frac{2ib^2\bar{p}}{\pi}\sum_{m=2}^{\infty}\frac{1}{m^2-1}\cos m\theta\\ &\quad = A_0+C_0\cosh\alpha_0(l/R)+(A_1+C_1\cosh\alpha_1(l/R))\cos\theta\\ &\quad\quad+\sum_{m=2}^{\infty}[A_m\cosh\beta_m(l/R)+C_m\cosh\alpha_m(l/R)]\cos m\theta,\end{aligned}\tag{23}$$

$$\begin{aligned}&\frac{\bar{p}}{2\pi}+\left[\frac{\bar{p}(l^2/R^2-2)}{2\alpha_1^2\pi}+\frac{\bar{p}}{\pi}\right]\cos\theta+\frac{\bar{p}}{\pi}\sum_{m=2}^{\infty}\frac{1}{m^2-1}\cos m\theta\\ &\quad = \frac{i}{2b^2}C_0\alpha_0^2\cosh\alpha_0(l/R)+\frac{i}{2b^2}[-A_1+(\alpha_1^2-1)C_1\cosh\alpha_1(l/R)]\cos\theta\\ &\quad\quad+\frac{i}{2b^2}\sum_{m=2}^{\infty}[(\beta_m^2-m^2)A_m\cosh\beta_m(l/R)+(\alpha_m^2-m^2)C_m\cosh\alpha_m(l/R)]\cos m\theta,\end{aligned}\tag{24}$$

from which one finds

$$\left.\begin{aligned} C_0 &= -\frac{ib^2\bar{p}}{\pi\alpha_0^2\cosh\alpha_0(l/R)}, \\ A_0 &= \frac{ib^2\bar{p}}{\pi}\left(1+\frac{1}{\alpha_0^2}\right), \\ C_1 &= \frac{2ib^2\bar{p}(1-\alpha_1^2)}{\pi\alpha_1^4\cosh\alpha_1(l/R)}, \\ A_1 &= \frac{ib^2\bar{p}[\alpha_1^2(2+l^2/R^2)-2]}{\pi\alpha_1^4}, \\ C_m &= \frac{2ib^2\bar{p}(1+\beta_m^2-m^2)}{\pi(m^2-1)(\alpha_m^2-\beta_m^2)\cosh\alpha_m(l/R)}, \\ A_m &= \frac{-2ib^2\bar{p}(1+\alpha_m^2-m^2)}{\pi(m^2-1)(\alpha_m^2-\beta_m^2)\cosh\beta_m(l/R)}. \end{aligned}\right\} \tag{25}$$

Finally one must ensure the conditions of continuity and symmetry, at $\theta = \pi$, for the stresses and displacements resulting from the particular integral ψ. We denote these quantities by w^*, T_ξ^*, $T_\theta^*, \ldots$, etc. Since ψ is an even function of θ, the only conditions that must be satisfied are

$$\left.\begin{aligned} &\left(\text{All odd functions such as } N_\theta^*, v^*, \frac{\mathrm{d}w^*}{\mathrm{d}\theta}, \ldots, \text{etc.}\right) = 0 \\ &\frac{\mathrm{d}}{\mathrm{d}\theta}(\text{All even functions such as } w^*, T_\xi^*, T_\theta^*, \ldots, \text{etc.}) = 0 \end{aligned}\right\} \text{at } \theta = \pi. \tag{26}$$

Observing the structure of quantities like $N_\theta^*, \ldots$, etc., and others such as $T_\xi^*, \ldots$, etc., it is seen that the former and the latter are respectively formed from odd and even derivatives of ψ. Since all odd derivatives of ψ are zero at $\theta = \pi$, the conditions (26) are therefore satisfied. It should be noticed that $\tilde{S}^*$ is at most a constant which must be set equal to zero in order to meet conditions $T_{\theta\xi}^* = M_{\theta\xi}^* = 0$ at $\theta = \pi$.

In practical cases, distributed loads are more common than line loads. However the solution to the problem of a distributed load, being constant along the length of the shell and varying in the circumferential direction can be found from (20), since $-\tilde{T}/\bar{p}$ is the Green's function for problems of this type. Therefore, for the solution $\tilde{T}_d$ of a distributed load, one has

$$\left.\begin{aligned} \tilde{T}_d &= -\frac{R}{\bar{p}}\int_{-\pi}^{-\pi+\theta}\tilde{T}([2\pi+\tau-\theta],\xi)p(\tau)\,\mathrm{d}\tau - \frac{R}{\bar{p}}\int_{-\pi+\theta}^{\pi}\tilde{T}([\theta-\tau],\xi)p(\tau)\,\mathrm{d}\tau \ \text{ for } \pi \geq \theta \geq 0 \\ T_d &= -\frac{R}{\bar{p}}\int_{-\pi}^{\pi+\theta}\tilde{T}([\tau-\theta],\xi)p(\tau)\,\mathrm{d}\tau - \frac{R}{\bar{p}}\int_{\pi+\theta}^{\pi}\tilde{T}([2\pi-\tau+\theta],\xi)p(\tau)\,\mathrm{d}\tau \ \text{ for } 0 \geq \theta \geq -\pi \end{aligned}\right\} \tag{27}$$

where $p(\theta)$ is the radially distributed load per unit area of the middle surface, and τ is the dummy variable of integration.

4. CONVERGENCE

Let us analyze the behavior of A_m and C_m and related terms for sufficiently large m. Squaring α_m and β_m, given by relations (19), one finds

$$\left.\begin{aligned} \alpha_m^2 &= m^2+\gamma-i(\lambda+b^2), \\ \beta_m^2 &= m^2-\gamma+i(\lambda-b^2), \\ \alpha_m^2 - m^2+1 &= \gamma+1-i(\lambda+b^2), \\ \beta_m^2-m^2+1 &= -\gamma+1+i(\lambda-b^2), \\ \alpha_m^2-\beta_m^2 &= 2(\gamma-i\lambda). \end{aligned}\right\} \tag{28}$$

Furthermore, factoring out m in the expressions for γ and λ, given by (19), we also obtain

$$\left.\begin{aligned} \gamma &= m\left\{\frac{1}{2}\left(1-\frac{b^4}{m^2}\right)+\frac{1}{2}\left[\left(1-\frac{b^4}{m^2}\right)^2+4b^4\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}, \\ \lambda &= m\left\{-\frac{1}{2}\left(1-\frac{b^4}{m^2}\right)+\frac{1}{2}\left[\left(1-\frac{b^4}{m^2}\right)^2+4b^4\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}. \end{aligned}\right\} \tag{29}$$

It is clear from (29) that for large m, γ and λ tend to the following expressions

$$\left.\begin{aligned} \gamma &\sim mk, \\ \lambda &\sim mq, \end{aligned}\right\} \tag{30}$$

in which

$$k = [\tfrac{1}{2}+\tfrac{1}{2}(1+4b^4)^{\frac{1}{2}}]^{\frac{1}{2}} = \text{constant},$$
$$q = [-\tfrac{1}{2}+\tfrac{1}{2}(1+4b^4)^{\frac{1}{2}}]^{\frac{1}{2}} = \text{constant}.$$

Also from relations (28) we find for large m

$$\left.\begin{aligned} \alpha_m^2-m^2+1 &\sim \gamma-i\lambda, \qquad & \alpha_m^2-m^2 &\sim \gamma-i\lambda, \\ \beta_m^2-m^2+1 &\sim -\gamma+i\lambda, \qquad & \beta_m^2-m^2 &\sim -\gamma+i\lambda. \end{aligned}\right\} \tag{31}$$

Substituting relations (31) and the last of (28) into the expressions for C_m and A_m, given by (25), and their products by $(\alpha_m^2-m^2)$ and $(\beta_m^2-m^2)$ one obtains

$$\left.\begin{aligned} C_m &\sim \frac{-ib^2\bar{p}}{\pi m^2\cosh\alpha_m(l/R)}, \qquad & (\alpha_m^2-m^2)C_m &\sim \frac{-ib^2\bar{p}(k-iq)}{\pi m\cosh\alpha_m(l/R)}, \\ A_m &\sim \frac{-ib^2\bar{p}}{\pi m^2\cosh\beta_m(l/R)}, \qquad & (\beta_m^2-m^2)A_m &\sim \frac{ib^2\bar{p}(k-iq)}{\pi m\cosh\beta_m(l/R)}. \end{aligned}\right\} \tag{32}$$

It remains to examine the behavior of α_m and β_m for sufficiently large m. To do this let us factor out $(m^2+\gamma)$ and $(m^2-\gamma)$ respectively in the expressions for α_m and β_m, given by (19), to get

$$\left.\begin{aligned}\alpha_m &= \left\{\tfrac{1}{2}(m^2+\gamma)+\tfrac{1}{2}(m^2+\gamma)\left[1+\left(\frac{\lambda+b^2}{m^2+\gamma}\right)^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}\\&\quad -i\left\{-\tfrac{1}{2}(m^2+\gamma)+\tfrac{1}{2}(m^2+\gamma)\left[1+\left(\frac{\lambda+b^2}{m^2+\gamma}\right)^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}},\\ \beta_m &= \left\{\tfrac{1}{2}(m^2-\gamma)+\tfrac{1}{2}(m^2-\gamma)\left[1+\left(\frac{\lambda-b^2}{m^2-\gamma}\right)^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}\\&\quad +i\left\{-\tfrac{1}{2}(m^2-\gamma)+\tfrac{1}{2}(m^2-\gamma)\left[1+\left(\frac{\lambda-b^2}{m^2-\gamma}\right)^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}.\end{aligned}\right\} \tag{33}$$

Since, according to expressions (30), γ and λ behave as m the terms $(\lambda+b^2)/(m^2+\gamma)$ and $(\lambda-b^2)/(m^2-\gamma)$ tend to zero when m becomes large, and hence one could use the binomial expansion to get

$$\left.\begin{aligned}\alpha_m &= \left\{\tfrac{1}{2}(m^2+\gamma)+\tfrac{1}{2}(m^2+\gamma)\left[1+\frac{1}{2}\left(\frac{\lambda+b^2}{m^2+\gamma}\right)^2-\frac{1}{8}\left(\frac{\lambda+b^2}{m^2+\gamma}\right)^4+\dots\right]\right\}^{\frac{1}{2}}\\&\quad -i\left\{-\tfrac{1}{2}(m^2+\gamma)+\tfrac{1}{2}(m^2+\gamma)\left[1+\frac{1}{2}\left(\frac{\lambda+b^2}{m^2+\gamma}\right)^2-\frac{1}{8}\left(\frac{\lambda+b^2}{m^2+\gamma}\right)^4+\dots\right]\right\}^{\frac{1}{2}}\\&\sim m-i\frac{q}{2},\\ \beta_m &= \left\{\tfrac{1}{2}(m^2-\gamma)+\tfrac{1}{2}(m^2-\gamma)\left[1+\frac{1}{2}\left(\frac{\lambda-b^2}{m^2-\gamma}\right)^2-\frac{1}{8}\left(\frac{\lambda-b^2}{m^2-\gamma}\right)^4+\dots\right]\right\}^{\frac{1}{2}}\\&\quad +i\left\{-\tfrac{1}{2}(m^2-\gamma)+\tfrac{1}{2}(m^2-\gamma)\left[1+\frac{1}{2}\left(\frac{\lambda-b^2}{m^2-\gamma}\right)^2-\frac{1}{8}\left(\frac{\lambda-b^2}{m^2-\gamma}\right)^4+\dots\right]\right\}^{\frac{1}{2}}\\&\sim m+i\frac{q}{2}.\end{aligned}\right\} \tag{34}$$

Substituting the above results into (32) we finally obtain

$$\left.\begin{aligned}C_m &\sim \frac{2b^2\bar{p}}{\pi m^2\,\mathrm{e}^{ml/R}}\left(\sin\frac{ql}{2R}-i\cos\frac{ql}{2R}\right),\\ A_m &\sim -\frac{2b^2\bar{p}}{\pi m^2\,\mathrm{e}^{ml/R}}\left(\sin\frac{ql}{2R}+i\cos\frac{ql}{2R}\right),\\ (\alpha_m^2-m^2)C_m &\sim \frac{2b^2\bar{p}}{\pi m\,\mathrm{e}^{ml/R}}\left[k\sin\frac{ql}{2R}-q\cos\frac{ql}{2R}-i\left(q\sin\frac{ql}{2R}+k\cos\frac{ql}{2R}\right)\right],\\ (\beta_m^2-m^2)A_m &\sim \frac{2b^2\bar{p}}{\pi m\,\mathrm{e}^{ml/R}}\left[k\sin\frac{ql}{2R}+q\cos\frac{ql}{2R}-i\left(q\sin\frac{ql}{2R}-k\cos\frac{ql}{2R}\right)\right].\end{aligned}\right\} \tag{35}$$

Therefore it is seen that the series in $\tilde{T}$ converges as fast as $(m^2\,\mathrm{e}^{ml/R})^{-1}$ at $\xi = 0$. The series involved in $\tilde{T}_1$ and $\tilde{T}_2$, and hence those in T_ξ, T_θ, M_ξ, and M_θ converge not slower than $(m\,\mathrm{e}^{ml/R})^{-1}$ at the middle of the shell. It is interesting to note that for shells with $l/R \geq 2$ the magnitude of the terms in the series decreases rapidly even at the beginning of the series, as will be seen in the numerical results.

5. AN APPROXIMATE CLOSED FORM SOLUTION FOR COMPARATIVELY LONG SHELLS

A review of C_0, C_1, C_m, and A_m shows that, for a fixed m, the absolute values of these quantities decay exponentially as l/R becomes large. On the other hand, A_1 grows with $(l/R)^2$ while A_0 remains constant. Therefore, for a finite ξ the quantities involving C_0, C_1, C_m, and A_m tend to zero as l/R becomes infinitely large, and one could write the expression (20) as

$$\left.\begin{aligned} \tilde{T} &= -\varepsilon(\theta)b^2\bar{p}i\sin\theta + \frac{ib^2\bar{p}}{2\pi}(\cos\theta + 2\theta\sin\theta) + A_0 + \left(A_1 - \frac{ib^2\bar{p}}{\alpha_1^2\pi}\xi^2\right)\cos\theta, \\ &\text{for } l/R \to \infty, \qquad \xi = \text{a finite number}, \qquad \pi \geq \theta \geq -\pi. \end{aligned}\right\} \tag{36}$$

Similar expressions for $\tilde{T}_1$ and $\tilde{T}_2$ and the other quantities can also be written. Physically this means that the edge effect, represented by the terms involving C_0, C_1, C_m, and A_m, decays far away from the edges which, in this case, are located at infinity. For comparatively long shells the expression (36) can be used as an approximate solution for the region in which $|\xi| \ll l/R$, since the terms in the series become negligible as compared to those given by (36). The minimum value of the ratios l/R for which formula (36) could be used as an approximation is discussed in the conclusion.

6. NUMERICAL RESULTS

In the following, we give numerical results for the cases in which $b = 5{\cdot}5$ corresponding to $R/t = 18{\cdot}308$, $\nu = 0{\cdot}3$, and l/R range from 1 to 15. The values of α_J and β_J for J ranging from zero to five are listed in Table 1. The corresponding values of A_J and C_J for various l/R are given in Tables 2 through 4. The computed constants A_J and C_J are then inserted into

TABLE 1. VALUES OF ROOTS OF CHARACTERISTIC EQUATIONS FOR $b = 5{\cdot}5$

J	α_J	β_J
0	$5{\cdot}5(1-i)$	—
1	$5{\cdot}5916 - 5{\cdot}4099i$	—
2	$5{\cdot}8818 - 5{\cdot}1595i$	$0{\cdot}3328 + 0{\cdot}2920i$
3	$6{\cdot}3938 - 4{\cdot}8156i$	$0{\cdot}8468 + 0{\cdot}6378i$
4	$7{\cdot}1079 - 4{\cdot}4716i$	$1{\cdot}5615 + 0{\cdot}9824i$
5	$7{\cdot}9572 - 4{\cdot}1859i$	$2{\cdot}4111 + 1{\cdot}2684i$

(21) and (22) to obtain, with the use of (3), the dimensionless quantities $T_\xi/\bar{p}$, $T_\theta/\bar{p}$, $M_\xi/R\bar{p}$, and $M_\theta/R\bar{p}$ at $\xi = \theta = 0$. These are given in Table 5. The longitudinal and circumferential

Table 2. Values of constants $A_0/\bar{p}$, $C_0/\bar{p}$, $A_1/\bar{p}$ and $C_1/\bar{p}$ for $b = 5{\cdot}5$, $\nu = 0{\cdot}3$ and various l/R

l/R	$A_0/\bar{p}$	$C_0/\bar{p}$	$A_1/\bar{p}$	$C_1/\bar{p}$
1	$-0{\cdot}1592+9{\cdot}6289i$	$(0{\cdot}9219-0{\cdot}9178i)\times 10^{-3}$	$-0{\cdot}4766+0{\cdot}0210i$	$(-0{\cdot}1612-0{\cdot}174i)\times 10^{-2}$
2	$-0{\cdot}1592+9{\cdot}6289i$	$(0{\cdot}2337-53{\cdot}16i)\times 10^{-7}$	$-0{\cdot}9535+0{\cdot}0368i$	$(-0{\cdot}1977-0{\cdot}8621i)\times 10^{-5}$
3	$-0{\cdot}1592+9{\cdot}6289i$	$(-0{\cdot}1526-0{\cdot}1546i)\times 10^{-7}$	$-1{\cdot}7484+0{\cdot}0631i$	$(-0{\cdot}2937-0{\cdot}1499i)\times 10^{-7}$
5	$-0{\cdot}1592+9{\cdot}6289i$	$(-0{\cdot}2594+0{\cdot}2537i)\times 10^{-12}$	$-4{\cdot}2921+0{\cdot}1471i$	$(-0{\cdot}1338+0{\cdot}4386i)\times 10^{-12}$
8	$-0{\cdot}1592+9{\cdot}6289i$	$(0{\cdot}2476+0{\cdot}0044i)\times 10^{-19}$	$-10{\cdot}4924+0{\cdot}3521i$	$(0{\cdot}1735-0{\cdot}1626i)\times 10^{-19}$
15	$-0{\cdot}1592+9{\cdot}6289i$	$(0{\cdot}3231+0{\cdot}3454i)\times 10^{-36}$	$-36{\cdot}0884+1{\cdot}1983i$	$(0{\cdot}1995-0{\cdot}1316i)\times 10^{-36}$

Table 3. Values of constants $A_2/\bar{p}$, $A_3/\bar{p}$, $A_4/\bar{p}$ and $A_5/\bar{p}$ for $b = 5{\cdot}5$, $\nu = 0{\cdot}3$ and various l/R

l/R	$A_2/\bar{p}$	$A_3/\bar{p}$	$A_4/\bar{p}$	$A_5/\bar{p}$
1	$-0{\cdot}9007-6{\cdot}2014i$	$-1{\cdot}0260-1{\cdot}5430i$	$-0{\cdot}4420-0{\cdot}2180i$	$-0{\cdot}1206-0{\cdot}0126i$
2	$-2{\cdot}3362-5{\cdot}2986i$	$-0{\cdot}8556-0{\cdot}1736i$	$-0{\cdot}0860+0{\cdot}0547i$	$(-0{\cdot}4224+0{\cdot}9946i)\times 10^{-2}$
3	$-3{\cdot}3772-3{\cdot}3602i$	$-0{\cdot}3275+0{\cdot}1626i$	$(-0{\cdot}489\ +21{\cdot}34i)\times 10^{-3}$	$(0{\cdot}739+0{\cdot}627i)\times 10^{-3}$
5	$-2{\cdot}4918-0{\cdot}1774i$	$0{\cdot}0110+0{\cdot}0660i$	$(0{\cdot}876\ -0{\cdot}340i)\times 10^{-3}$	$(-0{\cdot}2-0{\cdot}8i)\times 10^{-5}$
8	$-0{\cdot}6075+0{\cdot}6482i$	$0{\cdot}0046-0{\cdot}0025i$	$(-0{\cdot}009\ +0{\cdot}001i)\times 10^{-3}$	$(0{\cdot}4501+3390i)\times 10^{-8}$
15	$0{\cdot}0830+0{\cdot}0243i$	$(0{\cdot}3562+1{\cdot}3601i)\times 10^{-5}$	$(-0{\cdot}1571+0{\cdot}9156i)\times 10^{-10}$	$(-0{\cdot}0976-0{\cdot}2449i)\times 10^{-15}$

Table 4. Values of constants $C_2/\bar{p}$, $C_3/\bar{p}$, $C_4/\bar{p}$ and $C_5/\bar{p}$ for $b = 5{\cdot}5$, $\nu = 0{\cdot}3$ and various l/R

l/R	$C_2/\bar{p}$	$C_3/\bar{p}$	$C_4/\bar{p}$	$C_5/\bar{p}$
1	$(0{\cdot}4337-1{\cdot}6837i)\times 10^{-3}$	$(-0{\cdot}2904-0{\cdot}9164i)\times 10^{-3}$	$(-0{\cdot}3078-0{\cdot}2571i)\times 10^{-3}$	$(-0{\cdot}1354-0{\cdot}0310i)\times 10^{-3}$
2	$(-0{\cdot}3712-0{\cdot}3122i)\times 10^{-5}$	$(-0{\cdot}1574+0{\cdot}0325i)\times 10^{-5}$	$(-0{\cdot}1443+0{\cdot}2949i)\times 10^{-6}$	$(0{\cdot}1444+0{\cdot}4646i)\times 10^{-7}$
3	$(-0{\cdot}1233+0{\cdot}0557i)\times 10^{-7}$	$(0{\cdot}0269+0{\cdot}2674i)\times 10^{-8}$	$(0{\cdot}2626+0{\cdot}0572i)+10^{-9}$	$(0{\cdot}1152-0{\cdot}1254i)\times 10^{-10}$
5	$(0{\cdot}0939+0{\cdot}0477i)\times 10^{-12}$	$(0{\cdot}0796-0{\cdot}7470i)\times 10^{-14}$	$(-0{\cdot}1737+0{\cdot}0475i)\times 10^{-15}$	$(0{\cdot}0637+0{\cdot}1989i)\times 10^{-17}$
8	$(-0{\cdot}2221-0{\cdot}0545i)\times 10^{-20}$	$(0{\cdot}3212+0{\cdot}1418i)\times 10^{-22}$	$(-0{\cdot}8256-0{\cdot}5426i)\times 10^{-25}$	$(0{\cdot}2807+0{\cdot}8512i)\times 10^{-28}$
15	$(-0{\cdot}0686+0{\cdot}2945i)\times 10^{-38}$	$(-0{\cdot}1172+0{\cdot}0541i)\times 10^{-41}$	$(-0{\cdot}2152-0{\cdot}1085i)\times 10^{-46}$	$(0{\cdot}3750-0{\cdot}4376i)\times 10^{-52}$

TABLE 5. NON-DIMENSIONAL VALUES OF RESULTANT FORCES AND MOMENTS FOR $b = 5{\cdot}5$, $\nu = 0{\cdot}3$ AND VARIOUS l/R

l/R	$T_\xi/\bar{p}$	$T_\theta/\bar{p}$	$M_\xi/R\bar{p}$	$M_\theta/R\bar{p}$
1	−2·3056	−0·8687	−0·0370	−0·1021
2	−3·8278	−0·5668	−0·0489	−0·1461
3	−5·1991	−0·4129	−0·0598	−0·1837
5	−6·7006	−0·2306	−0·0745	−0·2366
8	−11·0658	−0·1887	−0·0791	−0·2527
15	−35·9472	−0·2174	−0·0846	−0·2520

stresses σ_ξ and σ_θ at $\xi = \theta = 0$, $z = t/2$ are calculated from the following expressions

$$\left.\begin{aligned} \frac{t}{\bar{p}}\sigma_\xi &= \frac{T_\xi}{\bar{p}}+\frac{6R}{t}\frac{M_\xi}{R\bar{p}}, \\ \frac{t}{\bar{p}}\sigma_\theta &= \frac{T_\theta}{\bar{p}}+\frac{6R}{t}\frac{M_\theta}{R\bar{p}}. \end{aligned}\right\} \tag{37}$$

The results of the calculations are listed in Table 6.

TABLE 6. NON-DIMENSIONAL VALUES OF LONGITUDINAL AND CIRCUMFERENTIAL STRESSES AT $\xi = 0$, $\theta = 0$, $z = t/2$ FOR $b = 5{\cdot}5$, $\nu = 0{\cdot}3$ AND VARIOUS l/R

l/R	$(t/\bar{p})\sigma_\xi$	$(t/\bar{p})\sigma_\theta$
1	−6·3672	−12·0873
2	−9·1964	−16·6130
3	−11·7647	−20·5925
5	−14·8891	−26·2174
8	−19·7509	−27·9504
15	−45·2383	−27·9038

Except for the case $l/R = 1$, in which six terms in the series are taken, the rest of the above quantities are computed by keeping only four terms in the series.

Finally the values of the stresses σ_ξ and σ_θ at $\xi = \theta = 0$, $z = t/2$, calculated from relation (36), are given in Table 7 in order that one can compare them with those given in Table 6, and establish the range of validity of (36). The results given in Tables 6 and 7 are

TABLE 7. NON-DIMENSIONAL VALUES OF LONGITUDINAL AND CIRCUMFERENTIAL STRESSES, CALCULATED FROM FORMULA (36), AT $\xi = 0$, $\theta = 0$, $z = t/2$ FOR $b = 5{\cdot}5$, $\nu = 0{\cdot}3$ AND VARIOUS l/R

l/R	$(t/\bar{p})\sigma_\xi$	$(t/\bar{p})\sigma_\theta$
1	−8·2930	−26·4841
2	−8·7888	−26·5025
3	−9·6152	−26·5331
5	−12·2595	−26·6309
8	−18·7051	−26·8694
15	−45·3137	−27·8541

plotted in Fig. 1. It is seen that the relative errors made by employing the expressions (36) are 5% and 0·2% respectively for $l/R = 8$ and $l/R = 15$.

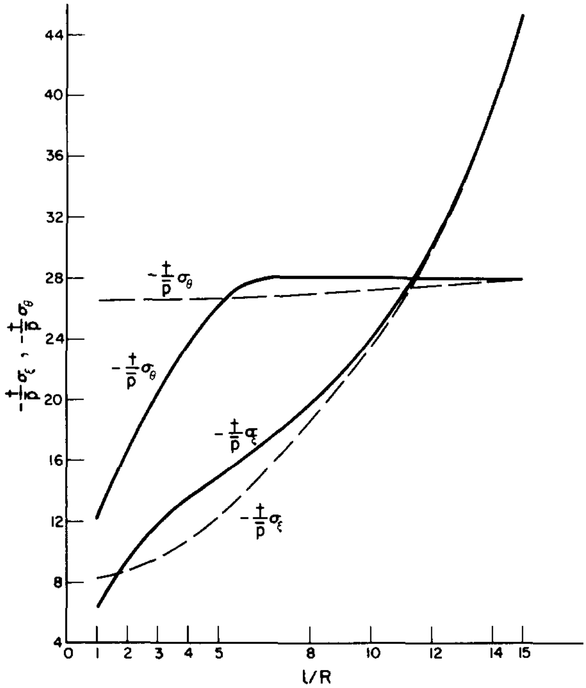

FIG. 1. Dimensionless longitudinal and circumferential stresses vs. l/R. The dash lines represent the stresses calculated from formula (36).

7. DISCUSSION

The minimum value of the ratios l/R for which formula (36) could be used as an approximate solution depends somewhat on the accuracy desired. For most engineering problems, in which 5% error is allowed, one could use the expression (36) for $l/R \geq 8$. Obviously, for better accuracy, one has to take a higher limit for l/R. When $l/R < 8$ we must take a few terms in the series. Again the number of these terms depends on the accuracy needed. For example, the above numerical results show that for $l/R = 5$ the absolute values of successive A_J's decrease at an average rate of about 1/70 for the first four terms in the series. Therefore for engineering problems one could achieve the desired accuracy by keeping the first two terms in the series, when $l/R = 5$.

Although the exact theoretical line loading of a cylindrical shell cannot be achieved in practice, one could find cases in which the load is distributed along a very narrow strip. As an example, consider a plate with a constant distributed load on the top, supported by several hollowed simply supported cylinders. On the other hand cylindrical shells, carrying a distributed load which is constant along the length of the shell, are of common occurrence in engineering. Among these one could mention simply supported horizontal tanks, filled completely or partially with fluid, horizontal pipes carrying fluid, and horizontal rotating cylindrical containers used in the chemical industry. Problems arising in these cases can be solved with the aid of (27).

Acknowledgement—The author is indebted to Dr. Ronald D. Palamara for invaluable editorial suggestions.

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(*Received* 24 *July* 1967; *revised* 18 *March* 1968)

Абстракт—Используется комплексное дифференциальное уравнение цилиндрической оболочки, полученное Новожиловым [1], для определения напряженного состояния, свободно опертой, замкнутой, цилиндрической оболочки, подверженной дойствию равномерной, внутренней радиальной линейной наргузке вдоль образующей. Задача определяется в замкнутом виде частного интнграла дифференциального уравнения, который выполняет краевые условия с помощью дополнительных решений, в форме одинарных рядов фурье. Эти рады сходятся оченв быстро в районах середины оболочки. Для сравнительно длинных оболочек выводятся приближенные выражение в замкнутом виде, для района очень удаленного от краев. Дается математическая попытка сходимости рядов и представляются численные результаты для некоторых отношений длины к радиусу.